JOURNAL OF ALGEBRA 10, 166–173 (1968)

Planar Algebraic Systems: Some Geometric Interpretations

MICHAEL ANSHEL AND JAMES R. CLAY*

University of Arizona, Tucson, Arizona 85721

Communicated by Marshall Hall, Jr.

Received January 2, 1968

Marshall Hall introduced planarity into algebra by coordinatizing projective planes with planar ternary rings [4]. In [9], J. L. Zemmer defines a planar near-field to be a near-field in which each equation $ax = bx + c$ has a unique solution for $a \neq b$. In our investigations of planarity, we discovered that if $(N, +, \cdot)$ is a near-ring satisfying the above equational property, then $(N, +, \cdot)$ is a near-field. (This was conjectured by both D. R. Hughes and J. L. Zemmer in private communications.) We shall see that the interaction of $+$ and $\cdot$ in the equation $ax = bx + c$ leads to the idea of a planar algebraic system. The algebraic structure of these systems will be discussed along with their geometric interpretation.

By a *left distributive system* we mean a triple $(N, +, \cdot)$ such that multiplication $\cdot$ is left distributive over addition $+$. Elements $a, b \in N$ are called *left equivalent multipliers*, denoted by $a \equiv {}_m b$, if and only if $ax = bx$ for all $x \in N$. The relation $\equiv_m$ is *discrete* when $a \equiv {}_m b$ implies $a = b$. A left distributive system is said to possess the *planar property* if the equation $ax = bx + c$ has a unique solution for $a \not\equiv {}_m b$.

DEFINITION 1. A left distributive system $(N, +, \cdot)$ with planar property is a *planar system* if

(1) in $(N, +)$ the right cancellation law is valid;

(2) in $(N, +)$ there is an identity 0;

(3) $(N, \cdot)$ is a semi-group;

(4) there are at least three points in N, no two of which are left equivalent multipliers.

A planar system is *integral* if 0 is the only left zero divisor.

Let $(N, +, \cdot)$ be an integral planar system. *Then* $0 \cdot x = x \cdot 0 = 0$ *for all*

* The second author received support from NSF contract GP-2141.

$x \in N$. The equation $a \cdot x = 0 \cdot x + 0$ has a unique solution, but if $z \in N$, then 0 and $0 \cdot z$ are solutions. Hence $0 \cdot z = 0$. (This proof, due to R. S. Pierce, is a considerable improvement over that of the authors.) Let N^* denote the nonzero elements of N. If $a \in N^*$, let 1_a denote the solution to the equation $ax = a$ and let $B_a = \{x \in N^* \mid x1_a = x\}$.

We can now give our main structure theorem.

THEOREM 1. *Let $(N, +, \cdot)$ be an integral planar system. Then*

(1) *each $(B_a, \cdot)$ is a group with identity 1_a;*

(2) *the family $\{B_a\}_{a\in N^*}$ is pairwise disjoint;*

(3) $N^* = \bigcup_{a\in N^*} B_a$;

(4) $N^*B_a = B_a$ *for each* $a \in N^*$;

(5) *if $a, c \in N^*$, then $\phi : B_a \to B_c$ defined by $\phi(x) = x1_c$ is a group isomorphism;*

(6) *each 1_a is a left identity for $(N, +, \cdot)$*

Proof. Let $a \in N^*$. Then $a \in B_a$, so $B_a \neq \emptyset$. If $x \in N^*$, and $b \in B_a$, then $xb \in N^*$ since N is an integral planar system. Now $(xb)\,1_a = x(b1_a) = xb$, so $xb \in B_a$. Hence (4) and B_a is closed under multiplication. Now $a = a1_a = (a1_a)\,1_a = a(1_a1_a)$, so $1_a1_a = 1_a$ by uniqueness of solutions to equation $ax = a$. Hence $1_a \in B_a$, so B_a has a right identity. Let $\bar{a}$ be the unique solution to the equation $ax = 1_a$. Then $a(\bar{a}1_a) = (a\bar{a})\,1_a = 1_a1_a = 1_a$, so $\bar{a}1_a = \bar{a}$ by uniqueness. Hence the right inverse of a is a member of B_a, so $(B_a, \cdot)$ is a group. This gives (1).

If $b \in B_a$, then $b1_a = b = b1_b$, hence $1_a = 1_b$ so $B_a = B_b$. This gives (2). From the definition of 1_a we get (3) immediately.

Let $\phi : B_a \to B_c$ be defined by $\phi(x) = x1_c$. Since (4) holds, we know ϕ is from B_a to B_c. If $x, y \in B_a$, then $\phi(xy) = (xy)\,1_c$ and $\phi(x)\,\phi(y) = (x1_c)(y1_c) = x(1_c(y1_c)) = x(y1_c) = (xy)\,1_c$ since $y1_c \in B_c$. So ϕ is a homomorphism. Now $b1_a = b$ if $b \in B_a$, so $b(1_a1_c) = (b1_a)\,1_c = b1_c$, hence $1_a1_c = 1_c$ follows from uniqueness. Suppose $x1_c = y1_c$ for $x, y \in B_a$. Then $(x1_c)\,1_a = x(1_c1_a) = x1_a = x$ and similarly, $x = (x1_c)\,1_a = (y1_c)\,1_a = y$, so ϕ is injective. If $y \in B_c$, then $y1_a \in B_a$. Hence $\phi(y1_a) = (y1_a)\,1_c = y$, so ϕ is surjective. This gives (5).

Define $D_a = \{x \in N^* \mid 1_ax = x\}$. We know $1_a0 = 0$, so we must show $D_a = N^*$ to prove (6). Let $b \in N^*$, so $b1_b = b$ and $1_b1_a = 1_a$ so $1_a \in D_b = \{x \in N^* \mid 1_bx = x\}$. If $x \in D_a$, then $1_ax = x$, so $1_bx = 1_b(1_ax) = (1_b1_a)\,x = 1_ax = x$. Hence $x \in D_b$ and $D_a \subseteq D_b$. Similarly, $D_b \subseteq D_a$. Since $b \in D_b$ and $a \in D_b$ we get $D_a = N^*$. This completes the proof of the theorem.

Note that (4) can be extended as follows. *If $S \subseteq N^*$ and if $N^*S \subseteq S$, then*

$S = \bigcup_{a \in T} B_a$ *for some* $T \subseteq N^*$. For if $a \in S$, then $xa \in S$ for all $x \in B_a$. Our assertion follows from the fact that $(B_a, \cdot)$ is a group.

COROLLARY. *Let* $(N, +, \cdot)$ *be a near-ring that is an integral planar system with* $\equiv_m$ *discrete. Then* $(N, +, \cdot)$ *is a planar near-field.*

Proof. If $a, b \in N^*$, then $1_a \equiv_m 1_b$, so $1_a = 1_b$. Hence $B_a = N^*$.

In the sequel a near-ring that is an integral planar system will be called an *integral planar near-ring*. From the corollary, note that if $(N, +, \cdot)$ is a near-ring in which each equation $ax = bx + c$ $(a \neq b)$ has a unique solution, then $(N, +, \cdot)$ is a near-field.

Let $(V, +)$ be a vector space of finite dimension $n \geqslant 2$ over a field F of characteristic $\neq 2$. Let $x = \langle x_1, \ldots, x_n \rangle \in V$. If $x = 0 \in V$, then set $\bar{x} = 0 \in F$. Otherwise set $\bar{x} = x_i$ where $x_i \neq 0$ but $x_1 = x_2 = \cdots = x_{i-1} = 0$. Define a multiplication $*$ on V by $a * b = \bar{a} \cdot b$ for $a, b \in V$ and where $\cdot$ denotes scalar multiplication. The following theorem gives nontrivial examples of integral planar near-rings.

THEOREM 2. *If* $(V, +)$ *is a vector space of dimension* $n \geqslant 2$ *over a field* F *of characteristic* $\neq 2$, *then*

(1) $(V, +, *)$ *is a planar near-rings*;

(2) $(V, +, *)$ *is a near-algebra over* F;

(3) *if* $\bar{B}_a = B_a \cup \{0\}$, then $(\bar{B}_a, +, *)$ *is a near-field and a left ideal of* $(V, +, *)$.

Proof. The proofs of (1) and (2) are direct. That each $(\bar{B}_a, +, *)$ is a near-field follows from a direct proof that $(\bar{B}_a, +)$ is a group and the fact that $(B_a, *)$ is a group (see(1) of Theorem 1). That $(V, +, *)$ is a left ideal follows from (4) of Theorem 1.

THEOREM 3. *Let* $(V, +)$ *be a normed linear space over the real numbers with norm* $\|\cdot\|$. *Define multiplication* $*$ *on* V *by* $a * b = \|a\|\, b$. *Then*

(1) $(V, +, *)$ *is a planar near-ring*;

(2) $(V, +, *)$ *is a near-algebra over the real numbers*;

(3) *if* $\bar{B}_a = B_a \cup \{0\} \cup B_{-a}$, *then* $(\bar{B}_a, +, *)$ *is an integral planar near-ring and a left ideal of* $(V, +, *)$.

Proof. The proofs of (1) and (2) are straightforward. If $a \neq 0$ and $a \in V$, then $1_a = a/\|a\|$. Hence $1_{-a} = -a/\|-a\| = -1_a$. Since $B_a = \{\lambda 1_a \mid \lambda > 0\}$, $B_{-a} = -B_a$. Now apply (1) and (4) of Theorem 1.

We remind the reader that an *affine plane* A is a triple (P, L, on) where P is a set of "points," L is a set of "lines" and *on* is a relation in $P \times L$ ($(p, l) \in P \times L$ is read p is *on* l or l is *on* p) satisfying: (a) two distinct points are *on* a unique line; (b) every line contains at least two distinct points; (c) there are at least 3 distinct points not *on* the same line; and (d) given a line $l \in L$ and a point $p \in P$, p not *on* l, there is one and only one line m *on* p such that l and m are not *on* any common point. Two lines $l, m \in L$ are said to be parallel, written $l \| m$, if and only if $l = m$ or l and m are *on* no common point. If $p, q \in P$, let $l(p, q)$ denote the line in L *on* both p and q. A map $\sigma : P \to P$ is called a *dilatation* if, and only if, for every pair $p, q \in P$, $p \neq q$, if $m \| l(p, q)$ where $\sigma(p)$ is *on* m then $\sigma(q)$ is *on* m. A map of all points in P onto a single point $p_0 \in P$ is called *degenerate*. All other dilatations are called *non-degenerate*. The mapping $I : P \to P$ defined by $I(p) = p$ for all $p \in P$ is called the *identity* map. A non-degenerate dilatation σ is called a *translation* if, and only if, either $\sigma = I$ or $\sigma(p) \neq p$ for all $p \in P$. An affine plane is said to be *coordinatized by a skew field* K if and only if there is a bijection $K \times K \to P$ such that the images of the set $\{(x, y) \mid y = xm + b\}$ and $\{(a, y) \mid y \in K\}$, for arbitrary but fixed m, $a \in K$, yield precisely the lines of the affine plane.

An affine plane $A = (P, L, on)$ can be coordinatized by a skew field if and only if the following axioms hold:

(4a) given any two points $p, q \in P$, there is a translation τ such that $\tau(p) = q$;

(4bP) for a given $p \in P$ and for any $q, r \in P$ such that p, q, r are distinct and collinear, there exists a dilation σ such that $\sigma(p) = p$ and $\sigma(q) = r$.

The translations of an affine plane form a group under the operation of composition. The group of translations is Abelian when axiom (4a) holds.

For more information on the above material see Chapter 2 in Artin [*1*]. The following concepts are discussed in a paper by V. P. Zarovnyi [7].

A group $(G, +)$ is a Φ (I, IV) group if and only if there is a family $\{G_\alpha\}$ of subgroups of G satisfying:

(1) $G = \bigcup_\alpha G_\alpha$;

(2) $G_\alpha \cap G_\beta = \{0\}$ if $\alpha \neq \beta$;

(3) each G_α is normal in G;

(4) if $\alpha \neq \beta$, then G_α and G_β generate G.

Theorems I, II, III, and IV of [7] show that if G is a Φ (I, IV) group, then the elements of G and the cosets of the G_α are the points and lines, respectively of an affine plane.

THEOREM 4. *A* Φ(I, IV) *group is Abelian and either torsion free or a direct sum of cyclic groups of prime order* p.

Proof. The relation $\|$ is an equivalence relation and the equivalence classes are called *pencils* of parallel lines. A non-degenerate dilatation preserves pencils. The pencils of a $\Phi(\mathrm{I}, \mathrm{IV})$ group G are the cosets of some G_α. The mappings $\tau_g : x \mapsto x + g$ permute the cosets of a G_α and therefore define a non-degenerate dilatation. Now τ_g is the identity or leaves no point fixed, so τ_g is a translation. Note that for $g_1, g_2 \in G$, there is a $g \in G$ such that $\tau_g(g_1) = g_2$. Hence axiom (4a) holds. Hence the group of translations is Abelian. But each translation τ is determined by $\tau(x)$ for some $x \in G$. So the group of translations is isomorphic to the group of mappings $\{\tau_g \mid g \in G\}$ which is isomorphic to G.

If there is an $x \in G$, $x \neq 0$, such that $nx = 0$ for some integer n, then $p \mid n$ for some prime p and $x_\alpha = (n/p)\, x$ has order p. Let $x_\alpha \in G_\alpha$. If $y \neq 0$ and $y \in G_\beta$, $\beta \neq \alpha$, then $p(x_\alpha + y) = py$. Now $x_\alpha + y \notin G_\alpha \cup G_\beta$ since $G_\alpha \cap G_\beta = \{0\}$. If $x_\alpha + y \in G_\gamma$, $\gamma \notin \{\alpha, \beta\}$, then $py = p(x_\alpha + y) \in G_\gamma$ yielding $py = 0$. Argue similarly with y replaced by x_α to show that any element in G_α has order p.

Note that the assumptions of the following theorem are satisfied by the examples of planar near-rings given in Theorem 2 for $n = 2$.

THEOREM 5. *Suppose $(N, +, \cdot)$ is an integral planar near-ring and each $\bar{B}_a = B_a \cup \{0\}$ is an additive normal subgroup. Also suppose that no $\bar{B}_a = N$ but any two $\bar{B}_a$, $\bar{B}_c$ generate N under $+$. Then:*

(1) *each $(\bar{B}_a, +, \cdot)$ is a near-field and a left ideal;*

(2) *$(\bar{B}_a, +, \cdot)$ is isomorphic to $(\bar{B}_c, +, \cdot)$ if $(x + y)\, 1_c = x 1_c + y 1_c$ for all $x, y \in B_a$;*

(3) *$(N, +)$ is Abelian and is isomorphic to the direct sum $\bar{B}_a \oplus \bar{B}_c$ as groups;*

(4) *the points of N are the points of an affine plane A with the cosets of the $\bar{B}_a$ as lines;*

(5) *the plane A can be coordinatized by a skew field.*

Proof. By hypothesis, each $(\bar{B}_a, +)$ is a group. Each $(B_a, \cdot)$ is a group follows from Theorem 1. Hence $(\bar{B}_a, +, \cdot)$ is a near-field. The mapping $\phi : \bar{B}_a \to \bar{B}_c$ defined by $\phi(x) = x 1_c$ is an isomorphism. Now (3) follows since $(N, +)$ is a $\Phi(\mathrm{I}, \mathrm{IV})$ group and the hypothesis that $\bar{B}_a$ and $\bar{B}_c$ generate $(N, +)$. Since $(N, +)$ is a $\Phi(\mathrm{I}, \mathrm{IV})$ group, (4) follows. There remains to show (5). In the proof of Theorem 4, we have shown that axiom (4a) is satisfied. The map $d_t : x \mapsto tx$, $t \neq 0$, is a dilatation fixing 0. To see this, note that $t\bar{B}_{x-y} \subseteq \bar{B}_{x-y}$ hence $t(\bar{B}_{x-y} + y) \subseteq \bar{B}_{x-y} + ty$ which implies that $tx, ty \in \bar{B}_{x-y} + ty$, a line parallel to $\bar{B}_{x-y} + y$. Let $0, x, y$ be distinct colinear points. Then $x, y \in \bar{B}_a$ for some a. Since $x \neq 0$ and $y \neq 0$, there is a $t \in B_a$ such that $tx = y$. Hence axiom (4bP) holds.

Theorem 3 provides examples of near-rings satisfying the conditions of the following theorem.

THEOREM 6. *Let* $(N, +, \cdot)$ *be an integral planar near-ring and define* $\bar{B}_a = B_a \cup \{0\} \cup B_{-a}$. *Suppose each* $\bar{B}_a$ *is an additive normal subgroup, no* $\bar{B}_a = N$, *but any two* $\bar{B}_a$, $\bar{B}_c$ *generate* N *under* $+$. *Then*

(1) *each* $(B_a, +, \cdot)$ *is an integral near-ring and a left ideal of* $(N, +, \cdot)$;

(2) $(\bar{B}_a, +, \cdot)$ *is isomorphic to* $(\bar{B}_c, +, \cdot)$ *if* $(x + y)\, 1_c = x1_c + y1_c$ *for all* $x, y \in \bar{B}_a$;

(3) $(N, +)$ *is Abelian and is isomorphic to the direct sum* $\bar{B}_a \oplus \bar{B}_c$ *as groups*;

(4) *the points of* N *are the points of an affine plane* A *with the cosets of the* $\bar{B}_a$ *as lines*;

(5) *the plane* A *can be coordinatized by a skew field provided each* $B_{-a} = -B_a$.

Proof. The proof of (1) follows from hypothesis and (4) of Theorem 1. The mapping $\phi : \bar{B}_a \to \bar{B}_c$ defined by $\phi(x) = x1_c$ is an isomorphism of near-rings. The group $(N, +)$ is a Φ(I, IV) group, hence (4) and (3) follows from Theorem 4 and hypothesis. There remains to show (5). The maps $\alpha : x \mapsto -x$ and $\beta : x \to tx$, $t \neq 0$, define dilatations. Suppose $x, y \in \bar{B}_a$, 0, x, and y distinct. If $x, y \in B_a$, choose $t \in B_a$ such that $tx = y$ and argue as in the proof of Theorem 5. Do similarly if $x, y \in B_{-a}$. Finally, suppose $x \in B_a$ and $y \in B_{-a}$. Then there is a t such that $t(-x) = y$. The dilatation $\beta \circ \alpha$ fixes 0 and sends x into y. Hence axiom (4bP) holds. The proof of Theorem 4 shows that axiom (4a) holds, hence our proof is complete.

THEOREM 7. *Suppose* $(N, +, \cdot)$ *is a finite integral planar near-ring and each* $\bar{B}_a = B_a \cup \{0\}$ *is an additive subgroup. Also suppose that no* $\bar{B}_a = N$ *but any two* $\bar{B}_a$, $\bar{B}_c$ *generate* N *under* $+$. *Then*

(1) $(N, +)$ *is Abelian*;

(2) *the points of* N *are the points of an affine plane* A *with the cosets of the* $\bar{B}_a$ *as lines*;

(3) *the affine plane* A *can be coordinatized by a field* $(F, +, \cdot)$;

(4) *each* $(\bar{B}_a, +, \cdot)$ *is a field*;

(5) *each* $\bar{B}_a = \{(x, mx) \mid x \in F\}$ *for some* $m \in F$, *or* $\bar{B}_a = \{(0, x) \mid x \in F\}$.

Proof. Each $(\bar{B}_a, +, \cdot)$ is a finite near-field, and Zassenhaus (in [8]) has shown that $(\bar{B}_a, +)$ is isomorphic to a direct sum of the cyclic groups Z_p, p a prime. Hence $(N, +)$ is a finite p-group. We know from Theorem 4.3.1 of [5] that the center Z of N is nontrivial.

Let $c \neq 0$ be in Z. If $a \neq 0$, there is a unique $x \in N$ such that $ax = c$; i.e., $f_a(x) = c$ where f_a is the automorphism of $(N, +)$ defined by $f_a(x) = ax$. Since f_a^{-1} is also an automorphism, we have $x = f_a^{-1}(c)$, so $x \in Z$. As c runs thru Z, so does x; hence $ax \in Z$ for all $x \in Z$.

If $a \in Z$ and $a \neq 0$, then $a \in B_a$ and there is an a' such that $a'a = 1_a$, so $1_a \in Z$. This gives $B_a \subseteq Z$. I.e., if $a \in Z$ and $a \neq 0$, then $B_a \subseteq Z$. Since $\bar{B}_a$, $\bar{B}_c$ generate N if $\bar{B}_a \neq \bar{B}_c$, we have for $x, y \in N$, $x = \alpha + \beta$ and $y = \alpha' + \beta'$ where $\alpha, \alpha' \in B_a$ and $\beta, \beta' \in B_c$. So

$$x + y = (\alpha + \beta) + (\alpha' + \beta') = (\alpha' + \beta') + (\alpha + \beta) = y + x.$$

Hence $(N, +)$ is Abelian. This gives (1).

The hypothesis of Theorem 5 are satisfied, hence (2). By (5) of Theorem 5, the affine plane can be coordinitized by a finite skew field, but a finite skew field is a field $(F, +, \cdot)$.

Now each $(\bar{B}_a, +, \cdot)$ is a near-field, so we need only show that $\cdot$ is commutative. We now view $(N, +)$ as the group of vectors located at $0 \in N$ associated with the affine plane A. The maps $t \mapsto s \cdot t$, $s \in B_a$, are dilatations of A and endomorphisms of $(\bar{B}_a, +)$, so they act as stretchings of our vectors. Hence they commute since the affine plane A is described by a vector space over a field. This gives (4).

Part (5) follows from (3).

We conclude with some amusing examples of integral planar systems:

(1) Let $(F, +, \cdot)$ be a field. Define $+_\lambda$ $(\lambda \neq 0)$ by

$$a +_\lambda b = \begin{cases} b, & \text{if } a = 0 \\ a + (\lambda b), & \text{if } a \neq 0. \end{cases}$$

Then $(F, +_\lambda, \cdot)$ is a nontrivial integral planar system where $\equiv_m$ is discrete and $+_\lambda$ is not necessarily associative.

(2) The following diagram defines a multiplication $\cdot$ on the cyclic group $(Z_5, +)$ such that $(Z_5, +, \cdot)$ is an integral planar near-ring. Note that $B_1 = \{1, 4\}$, $B_2 = \{2, 3\}$.

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	4	3	2	1
3	0	1	2	3	4
4	0	4	3	2	1

Define $\bar{B}_i = B_i \cup \{0\}$ and $B_{ij} = \bar{B}_i + j$, $i = 1, 2; j \in Z_5$. If we let $I = Z_5$, then the B_{ij} are circles of an inverse plane [*3*, *6*]. This example was obtained using a digital computer. (See [*2*]).

References

1. Artin, E. "Geometric Algebra." Wiley (Interscience), New York, 1957.
2. Clay, J. R. The near-rings on groups of low order. *Math. Z.* **104** (1968), 364-371.
3. Dembowski, P. and Hughes, D. R. On inversive planes. *J. London Math. Soc.* **40** (1965), 171–182.
4. Hall, M., Jr. Projective planes. *Trans. Am. Math. Soc.* **54** (1943), 229–277.
5. Hall, M., Jr. "The Theory of Groups". Macmillan, New York, 1959.
6. Hall, M., Jr. Group theory and block designs, *In* "Proceedings of the International Conference on Theory of Groups." Gordon and Breach, New York, 1967.
7. Zarovnyi, V. P. Interpretations of the plane axioms of affine geometry in an abstract group. *Ukrain. Math. Z.* **10** (1958), 351–364.
8. Zassenhaus, H. Uber endliche fastkorper. *Abhandl. Math. Sem. Hamburg* **11** (1936), 187–220.
9. Zemmer, J. L. Near-fields, planar and non-planar. *Math. Student* **31** (1964), 145–150.